Plan:

§1. p-adic Banach $^{(C-)}$ algebras.

§2. sympathetic alg.

§3. $\widetilde{C\{x\}}$ & $\widetilde{T}_C$ & stuff.

§1. fix p = prime. C = a cplt alg. closed NA extn of $\mathbb{Q}_p$ equipped w/ $|-|$, normalized $|p| = p^{-1}$.

Defn. $(\Lambda, \|-\|_\Lambda)$ is called <u>normed C-alg</u> if Λ is a C-alg. &

(below, our Λ is always assumed to be)

(0) $\|\lambda\| = 0 \Leftrightarrow \lambda = 0$.

(1) $\|c\lambda\| = |c| \cdot \|\lambda\|$

(2) $\|\lambda \cdot \lambda'\| \leq \|\lambda\| \cdot \|\lambda'\|$.

- $\mathcal{O}_\Lambda := \{\lambda \in \Lambda \mid \|\lambda\| \leq 1\}$, $m_\Lambda := \{\lambda \in \Lambda \mid \|\lambda\| < 1\}$, $\overline{\Lambda} := \mathcal{O}_\Lambda / m_\Lambda$.
- $|\Lambda|$ = norm set $= \{|\lambda| \in \mathbb{R}_{\geq 0} \mid \lambda \in \Lambda\} \subseteq \mathbb{R}_{\geq 0}$.
  (We shall always be in the case of $|C| = |\Lambda|$)

Terminology:
- $\|-\|$ is multiplicative if $\|\lambda \cdot \lambda'\| = \|\lambda\| \cdot \|\lambda'\|$.
  (when $|C| = |\Lambda|$, $\Leftrightarrow$ $\overline{\Lambda}$ is an integral domain).
- $(\Lambda, \|-\|_\Lambda)$ is called a Banach C-alg if it's cplt wrt $\|-\|_\Lambda$
- Given normed Λ, $\widehat{\Lambda}$:= cpltion of Λ, is a Banach C-alg

Defn. $\mathrm{Spec}(\Lambda) := \mathrm{Hom}_C(\Lambda, C)$ equipped w/ the (weak topology)
coarsest top. s.t. $\forall f \in \Lambda$, $\mathrm{Spec}(\Lambda) \longrightarrow C$, $s \longmapsto s(f) =: f(s)$
is continuous.

(open basis: $\{s \in \mathrm{Spec}(\Lambda) \mid |s(f_i) - x_i| < \varepsilon$ for some f_i & x_i, $1 \leq i \leq n$, $\varepsilon > 0\}$)

②

Exercise: • $(s,f) \longmapsto f(s) \;:\; \mathrm{Spec}(\Lambda) \times \Lambda \longrightarrow \mathbb{C}$ is continuous. (need a lemma later).

• $\mathrm{Spec}(\hat{\Lambda}) \longrightarrow \mathrm{Spec}(\Lambda)$ is a homeomorphism.

$(\Lambda, \|\|)$ is called <u>spectral</u> if

$$\|\lambda\| = \sup_{s \in \mathrm{Spec}(\Lambda)} |s(\lambda)| \quad \forall \lambda \in \Lambda.$$

Example: $\mathbb{C}\{X_1, \cdots, X_d\} := \left\{ \sum_{I \in \mathbb{N}^d} a_I X^I \in \mathbb{C}[\![\underline{X}]\!] \;\middle|\; |a_I| \to 0 \text{ when } |I| \to \infty \right\}.$

$$\|f\| := \max_{I \in \mathbb{N}^d} |a_I|.$$

Exercise: • $\mathbb{C}\{\underline{X}\}$ is a spectral Banach $\mathbb{C}$-alg.

• $\mathbb{C}[\underline{X}] \subseteq \mathbb{C}\{\underline{X}\}$ is dense.

Lemma: Λ_1, Λ_2 are normed $\mathbb{C}$-alg's, suppose Λ_2 satisfies $\|\lambda^n\|_{\Lambda_2} = \|\lambda\|_{\Lambda_2}^n$.
(if $|\Lambda_2| = |\mathbb{C}|$, then $\Leftrightarrow$ $\overline{\Lambda_2}$ is reduced).

then $\varphi: \Lambda_1 \longrightarrow \Lambda_2$ is continuous $\Leftrightarrow$ $\|\varphi(\lambda)\|_{\Lambda_2} \le \|\lambda\|_{\Lambda_1}$. (e.g. spectral alg.)

pf for $\Rightarrow$: ~~$\|\lambda\|_{\Lambda_1} \le A \cdot \|\varphi(\lambda)\|_{\Lambda_2}$, replace~~

suppose $\|\varphi(\lambda)\|_{\Lambda_2} \le A \cdot \|\lambda\|_{\Lambda_1}$, replace λ by λ^n. □

Exercise: $\mathrm{Spec}(\mathbb{C}\{\underline{X}\}) \longrightarrow \mathcal{O}_{\mathbb{C}}^d$ is a homeomorphism.

$$s \longmapsto (s(X_i))$$

If Λ is normed $\mathbb{C}$-alg, $\Lambda\{X\} := \left\{ \sum \lambda_i X^i \in \Lambda[\![X]\!] \;\middle|\; |\lambda_i| \xrightarrow[i \to \infty]{} 0 \right\}.$
$\|f\| := \max \|\lambda_i\|$.

Exercise: • Λ Banach $\Rightarrow$ $\Lambda\{X\}$ Banach.

• Λ spectral $\Rightarrow$ $\Lambda\{X\}$ spectral

• $\mathrm{Spec}(\Lambda\{X\}) = \mathrm{Spec}(\Lambda) \times \mathcal{O}_{\mathbb{C}}$.

• $\mathbb{C}\{\underline{X}\}\{X_{d+1}\} = \mathbb{C}\{X_1, \cdots, X_{d+1}\}$

Prop. (Weierstrass preparation)

⌈ Defn: $f = \sum \lambda_i X^i \in \mathcal{O}_\Lambda\{X\}$ ~~$\Lambda\{X\}$~~ is called regular of deg s, if f ~~g~~ $\in \Lambda[x]$ is a poly of deg s w/ unital leading coeff. ⌋

If $f \in \mathcal{O}_\Lambda\{X\}$ is regular of deg s, then $\exists!$ $g \in \mathcal{O}_\Lambda[X]$ of deg s and $u \in \Lambda\{X\}^*$ s.t. $f = u \cdot g$. (Cor: $\mathbb{C}\{X\}$ is a PID).

§2. Sympathetic alg.

- Λ is called conn'd if it doesn't contain nontrivial idempotent.
- Λ is conn'd $\Leftrightarrow$ $\Lambda\{X\}$ is conn'd.

~~§2. Sympathetic alg.~~

Notation: $\mathcal{O}_\Lambda^{**} := \{\lambda \in \mathcal{O}_\Lambda^* \mid \|\lambda - 1\| < 1\}$.

Exercise: ~~Show~~ When Λ is spectral Banach (or spectral colimit of Banach alg.), then

$$\mathcal{O}_\Lambda^{**} = \{\lambda \in \mathcal{O}_\Lambda \mid \|\lambda - 1\| < 1\}.$$

Defn. Λ is p-closed if $\mathcal{O}_\Lambda^{**} \xrightarrow{(-)^p} \mathcal{O}_\Lambda^{**}$.

Exercise: (1) $\varphi: \Lambda_1 \longrightarrow \Lambda_2$ w/ $\|\lambda^n\|_{\Lambda_2} = \|\lambda\|_{\Lambda_2}^n$ $\forall \lambda \in \Lambda_2$, $\varphi(\mathcal{O}_{\Lambda_1}^{**}) \subseteq \mathcal{O}_{\Lambda_2}^{**}$.

(2) If Λ is spectral, Λ is p-closed iff. $\forall \lambda \in \mathcal{O}_\Lambda^{**}$, "$\sqrt[p]{\lambda}$" exists in Λ (instead of $\mathcal{O}_\Lambda^{**}$).

Lemma: If Λ is spectral p-closed, then $\mathcal{O}_\Lambda / p\mathcal{O}_\Lambda \xrightarrow{(-)^p} \mathcal{O}_\Lambda / p\mathcal{O}_\Lambda$.

("spectral Banach p-closed C-alg's are perfectoid).

pf: let $a \in \mathcal{O}_\Lambda$. ~~solve~~ Let $b \in \mathcal{O}_\Lambda^{**}$ s.t. $b^p = 1 + \sqrt{p}\, a$.

then $(b-1)^p = b^p - 1 + \text{sth in } p\mathcal{O}_\Lambda$

$= \sqrt{p}\, a + \text{sth in } p\mathcal{O}_\Lambda$.

let $x = \text{"}\frac{1}{\sqrt[2p]{p}}\text{"} \cdot (b-1) \Rightarrow x^p = a + \text{sth in } \sqrt{p}\,\mathcal{O}_\Lambda \in \mathcal{O}_\Lambda$.

$\Rightarrow x \in \mathcal{O}_\Lambda$.

~~now let $z = x^p - a$~~.

start w/ $x^p = a + \sqrt{p}\, z$, we consider $y \in \mathcal{O}_\Lambda$.

$(x - \text{"}\sqrt[2p]{p}\text{"}\, y)^p \equiv x^p - \sqrt{p}\, y^p \mod p\mathcal{O}_\Lambda$.

since we can solve $y^p = z \mod \sqrt{p}\,\mathcal{O}_\Lambda$, we are done.

Exercise: If Λ is (1) p-closed normed C-alg.

(2) colim of Banach C-alg.

then $\hat{\Lambda}$ is p-closed.

Defn. Λ is called a <u>sympathetic alg</u> if it's a p-closed conn'd spectral Banach C-alg.

§3. $\widehat{C\{X\}}$, $\tilde{T}_C$ ...

Notation: $F := \mathrm{Frac}(C\{X\})$, equipped w/ the induced $\|-\|$
(as $\|-\|$ on $C\{X\}$ is multiplicative, actually it's even spectral

$\hat{F} :=$ completion. , $\overline{F}$ = alg. closure of F (fix one).

$\|-\|_{sp}$: a norm on $\overline{F}$ defined by:

$\forall\ x \in \overline{F}$, $\|x\|_{sp} = \max\{\|\hat{\varphi}(x)\|_{\overline{\hat{F}}} \mid \varphi \in \mathrm{Hom}_F(\overline{F}, \overline{\hat{F}})\}$

(Note that $\overline{\hat{F}}$ is equipped w/ a ~~unique~~ canonical norm, using Newton polygon ...)

Exercise: ① $\|x^n\|_{sp} = \|x\|_{sp}^n$, $\|xy\|_{sp} = \|x\|_{sp} \cdot \|y\|$, $\forall x \in \overline{F}$, $y \in F$.

② $\|-\|_{sp}$ is invariant under $\mathrm{Gal}(\overline{F}/F)$.

③ $\|x\|_{sp} \in |C|$.

④ if $X^n + a_{n-1}X^{n-1} + \dots + a_0$ is the minimal poly of x over F (so $a_i \in F$).
then $\|x\|_{sp} = \sup_{0 \le i \le n-1} \sqrt[n-i]{\|a_i\|}$.

More notations:

$$\begin{array}{ccc} \tilde{T}_C & \longrightarrow & \mathrm{Aut}(\overline{F}) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(C\{X\}) = B(0,1) = \mathcal{O}_C & \longrightarrow & \mathrm{Aut}(F) \\ a & \longmapsto & (X \mapsto X+a). \end{array}$$

pull back.

$$\rightsquigarrow \quad \mathrm{Gal}(\overline{F}/F) \longrightarrow \tilde{T}_C \longrightarrow \mathcal{O}_C$$

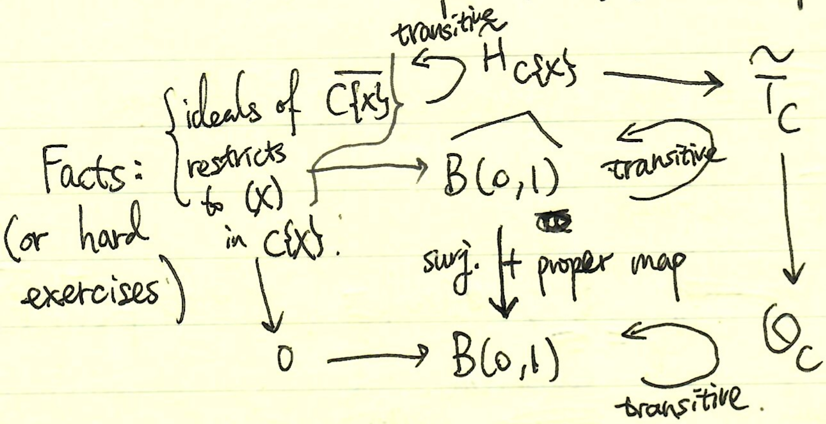
$$\tau \longmapsto x(\tau) := \tau(X) - X.$$

$\mathrm{Gal}(\overline{F}/F) = \tilde{H}_{C\{X\}}$.

$\overline{\mathbb{C}\{X\}}$:= int'l closure of $\mathbb{C}\{X\}$ in $\overline{F}$, equipped w/ $\|\cdot\|_{sp}$.

$\widehat{\mathbb{C}\{X\}}$:= completion. (Exercise : $\widetilde{T}_C \curvearrowright \overline{\mathbb{C}\{X\}}$ by isometry hence $\widehat{\mathbb{C}\{X\}}$ as well).

$$\widehat{B(0,1)} := \mathrm{Spec}(\overline{\mathbb{C}\{X\}}) = \mathrm{Spec}(\widehat{\mathbb{C}\{X\}}).$$



Notation: choose and fix $s_C \in \widehat{B(0,1)}$ above 0.

if $\tau \in \widetilde{T}_C$ & $f \in \widehat{\mathbb{C}\{X\}}$, $f(\tau) := s_C(\tau(f)) \in C$.
$= \tau(s_C)(f)$.

Prop. (1) $\widetilde{H}_{\mathbb{C}\{X\}} \times \widehat{B(0,1)} \times \widehat{\mathbb{C}\{X\}} \longrightarrow C$

$(\sigma, s, f) \longmapsto s(\sigma(f))$

is continuous.

(2) $f(\widetilde{H}_{\mathbb{C}\{X\}})$ is compact $\forall f \in \widehat{\mathbb{C}\{X\}}$.

pf : (1) ⇒ (2) as $\widetilde{H}_{\mathbb{C}\{X\}} = \mathrm{Gal}(\overline{F}/F)$ is profinite (⇒ cpt).

to show (1): say $x = s_0(\sigma_0(f_0))$, want to find nbhd in preim of $B(x, \varepsilon)$

then replace f_0 by approximation $f_1 \in \overline{\mathbb{C}\{X\}}$

an open subgp of $\widetilde{H}_{\mathbb{C}\{X\}}$ fixes f_1.

finally ~~$\widehat{B(0,1)}$~~ $\mathrm{Spec}(\Lambda) \times \Lambda \to C$ is continu

Technical exercise: We equip $\widetilde{T}_C$ w/ the topology s.t.

$$^a s_C : \widetilde{T}_C \twoheadrightarrow \widehat{B(0,1)} \quad \text{is a quotient map.}$$
$$\tau \longmapsto \tau(s_C)$$

(so topologize using $^a s_C^{-1}$(open)).

Then given $\{\tau_n\}$ a seq. in $\widetilde{T}_C$, if $\{x(\tau_n)\}$ converge, ($\Rightarrow \overline{\{x(\tau_n)\}}$ cpt)

$\{\tau_n\}$ have an accumulate pt in $\widetilde{T}_C$

($\Leftarrow$ $\{\tau_n(s_C)\}$ has — " — in $\widehat{B(0,1)}$
b/c $\widehat{B(0,1)} \twoheadrightarrow B(0,1)$ is proper).

Thm: $\widehat{C\{X\}}$ is sympathetic.

pf part I: • Banach ✓ by $\widehat{(\ )}$.
• p-closed ✓ by previous Exercise (on p. 4).

part II: show it's spectral:

Defn. $\Gamma_f \subseteq C \times C$

$$:= \{(x(\tau), f(\tau)) \mid \tau \in \widetilde{T}_C\} = \{(s(X), s(f)) \mid s \in \widehat{B(0,1)}\}$$

Prop: Let $f \in \widehat{C\{X\}}$ & $P = Y^n + \sum_{i=0}^{n-1} b_i Y^i \in C\{X\}[Y]$ its minimal poly. TFAE for $(x,y) \in B(0,1) \times C$:

(1) $\exists \tau \in \widetilde{T}_C$ s.t. $x(\tau) = x$ & $f(\tau) = y$

(2) $(x,y) \in \Gamma_f$

(3) $\exists s \in \widehat{B(0,1)}$ s.t. $s(X) = x$ & $s(f) = y$.

(4) $P(x,y) = 0$.

pf (1) – (3) obvious. (1)–(3) $\Rightarrow$ (4) obvious.

(4) $\Rightarrow$ rest: If $P(x,y) = 0$, then $C\{X\}[Y] \longrightarrow C$, $X, Y \longmapsto x, y$ factors thru

$$\mathbb{C}\{X\}[f] =: R = \mathbb{C}\{X\}[Y]/P(X,Y) \overset{\text{integral}}{\subseteq} \overline{\mathbb{C}\{X\}}.$$

$\Rightarrow$ the map $R \twoheadrightarrow \mathbb{C}$ extends to $\overline{\mathbb{C}\{X\}} \overset{\tilde{s}}{\twoheadrightarrow} \mathbb{C}$.

which is our desired $\tilde{s} \in \mathrm{Spec}(\overline{\mathbb{C}\{X\}}) = \widehat{B(0,1)}$.

(We used Exercise: $s: \overline{\mathbb{C}\{X\}} \to \mathbb{C}$ algebraic map is continuous $\iff \varphi|_{\mathbb{C}\{X\}}$ is continuous.)

pf: Newton polygon!

Cor: ① if $f \in \overline{\mathbb{C}\{X\}}$, then $\|f\|_{sp} = \sup_{\tau \in \tilde{T}_{\mathbb{C}}} |f(\tau)| = \sup_{s \in \widehat{B(0,1)}} |s(f)|$.

② : $\overline{\mathbb{C}\{X\}}$ & $\widehat{\mathbb{C}\{X\}}$ are spectral.

pf of ①: equality of latter is clear.

by previous Prop., we have

$$\sup_{s \in \widehat{B(0,1)}} |s(f)| = \sup_{x \in B(0,1)} \sup_{y \in \text{rts of } P(x,Y)} |y| = \sup_{x \in B(0,1)} \sup_{1 \le i \le n} \sqrt[i]{|b_{n-i}(x)|}$$

$$= \sup_{1 \le i \le n} \sqrt[i]{\|b_{n-i}\|} = \|f\|_{sp}.$$

① $\Rightarrow$ ② : clear.

part III: conn'd.

Lemma: If $f \in \overline{\mathbb{C}\{X\}}$ s.t. $f(id) = 0$ ($s_c(f) = 0$), then $\exists\ m < \deg(f / \mathbb{C}\{X\})$ & $\alpha_1, \cdots, \alpha_m \in B(0, \|f\|_{sp})$ s.t.

$$f(\tilde{T}_{\mathbb{C}}) \supseteq B(0, \|f\|_{sp}) \setminus \bigcup_{i=1}^{m} B(\alpha_i, \|f\|_{sp}^{-}).$$

Cor: ① If $f \in \widehat{\mathbb{C}\{X\}}$ satisfies $f(0) = 0$ (where $0 = \mathrm{id} \in \tilde{T}_c$) and $\exists$ $\rho \geq 0$ & S cpt s.t.

$$f(\tilde{T}_c) \subseteq S + B(0, \rho) \text{, then } \|f\|_{sp} \leq \rho.$$

② If $f \in \widehat{\mathbb{C}\{X\}}$ ~~has cpt~~ w/ $f(\tilde{T}_c)$ cpt, then $f \in \mathbb{C}$ (i.e. a constant).

pf: ① ⇒ ② immediate. Suppose $\rho < \|f\|_{sp}$.

pf of ①: choose $g \in \mathbb{C}\{X\}$ s.t. $\|f - g\|_{sp} < \|f\|_{sp}$.

replace g by $g - g(0)$ if necessary, we have:

$$g \in \mathbb{C}\{X\},\ \|g\|_{sp} = \|f\|_{sp},\ g(0) = 0,\ g(\tilde{T}_c) \subseteq S + B(0, \|g\|_{sp}^-).$$

BUT: previous lemma ⇒ $g(\tilde{T}_c)$ ~~infinite~~ mod $B(0, \|g\|_{sp}^-)$ is infinite!

Contradicting to S being cpt. (⇒ S mod $B(0, \|g\|_{sp}^-)$ is finite).

Cor. $\widehat{\mathbb{C}\{X\}}$ is conn'd b/c $\{0, 1\}$ is certainly cpt!

pf of lemma: WLOG, $\|f\|_{sp} = 1$. let $P(X, Y) \in \mathcal{O}_c\{X\}[Y]$ be minimal poly of f.

Note ~~that~~ by previous lemma: $y \in$ Image ⇔ $P(X, y) = 0$ has a solution.

Now we let $P(X, Y) = Y^n + a_{n-1}(X) \cdot Y^{n-1} + \cdots + a_0(X)$

w/ $a_i(X) = \sum_{k=0}^{\infty} a_{i,k} X^k$.

Expand in X instead: $(Y^n + a_{n-1,0} Y^{n-1} + \cdots + a_{0,0}) \cdot X^0$
$+ (a_{n-1,1} Y^{n-1} + \cdots + a_{0,1}) \cdot X + \cdots$

[Aside: given $h(X) = \sum_{i=0}^{\infty} b_i X^i \in \mathbb{C}\{X\}$, when does $h(X)=0$ have no root?

Answer: iff $|b_0| > |b_{>0}|$. ⌟

So ~~we want to make sure that~~ it suffices to find some k s.t. $|a_{n-1,k}\, y^{n-1} + \cdots + a_{0,k}| = 1$ for "most y" in $\mathcal{O}_{\mathbb{C}}$.

Claim: $\overline{P}(X,Y)$ in $k_{\mathbb{C}}[X,Y]$ cannot look like

$$Y^n + \overline{a_{n-1,0}}\, Y^{n-1} + \cdots + \overline{a_{m,0}}\, Y^m$$

for some $1 \le m \le n-1$! (and $\overline{a_{m,0}} \neq 0$).

pf: otherwise $P(X,Y)$ won't be irreducible (some kind of Hensel's lemma).

Note $f(0)=0$, therefore $a_{0,0} = P(0,0) = 0$.

So the above Claim + $\|f\|_{sp} = 1 \implies \exists\, k>0$ and some i s.t. $|a_{i,k}| = 1$.

For that particular k, let α_i be any lift of solutions of $\overline{a_{n-1,k}}\, y^{n-1} + \cdots + \overline{a_{0,k}} = 0$ in $k_{\mathbb{C}}$.

and we're done!